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## Dissipative quantum disordered models\*

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This article reviews recent studies of mean-field and one dimensional quantum disordered spin systems coupled to different types of dissipative environments. The main issues discussed are: (i) The real-time dynamics in the glassy phase and how they compare to the behaviour of the same models in their classical limit. (ii) The phase transition separating the ordered – glassy – phase from the disordered phase that, for some long-range interactions, is of second order at high temperatures and of first order close to the quantum critical point (similarly to what has been observed in random dipolar magnets). (iii) The static properties of the Griffiths phase in random Ising chains. (iv) The dependence of all these properties on the environment. The analytic and numeric techniques used to derive these results are briefly mentioned.

*Keywords:* Quantum spin models, disorder, glassiness

### 1. Introduction

Glasses slowly evolve towards equilibrium though never reaching it in observable times scales. Scientific research in this area started more than a century ago. The amount of experimental data gathered is huge. Systems that would reach equilibrium in observable time-scales, such as weakly sheared complex liquids, can be driven out of equilibrium by external perturbations and still evolve slowly. Powders stay in static metastable states unless externally tapped or sheared: these non-equilibrium perturbations slowly drive them towards more compact configurations. Even if *a priori* very different, these systems share many dynamic properties.

In several cases of practical interest quantum effects play an important role. On the experimental side spin-glass phases have been identified in many condensed matter systems at very low temperature. Among them we can cite the bi-layer Kagome system<sup>1</sup>  $\text{SrCr}_s\text{Ga}_4\text{O}_{19}$ , the polychlore structure<sup>2</sup>  $\text{Li}_x\text{Zn}_{1-x}\text{V}_2\text{O}_4$ , the dipolar magnet<sup>3,4</sup>  $\text{LiHo}_x\text{Y}_{1-x}\text{F}_4$  and the high  $T_c$  compound<sup>5</sup>  $\text{La}_{1-x}\text{Sr}_2\text{Cu}_2\text{O}_4$ . Quantum glassy phases exit also in electronic systems<sup>6</sup> and structural glasses<sup>7</sup> such as Mylar.

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The driven case is also very important in quantum systems, think of an electronic device driven by an external current.

The non-equilibrium dynamics of classical glasses, weakly driven complex liquids and granular matter have been rationalized within a theoretical approach that is based on the solution of mean-field simple models.<sup>8</sup> How much of the classical glassy phenomenology survives at very low temperatures where quantum effects are important is a question that deserves careful theoretical and experimental analysis. The impossibility of simulating the real-time evolution of quantum systems of moderate size enhances the importance of *solving* simple mean-field or low dimensional models.

On the other hand, peculiar phenomena in quantum phase transitions have been signaled analytically and experimentally in systems with and without quenched disorder.<sup>9,10</sup> For instance, at low temperatures and intermediate dipole concentration, the dipolar-coupled Ising magnet  $\text{LiHo}_x\text{Y}_{1-x}\text{F}_4$  in a transverse field exhibits a spin-glass-like phase.<sup>3,4</sup> The phase transition is of second order at low transverse field but becomes first order close to the quantum critical point.<sup>4</sup> Moreover, the relaxation in the glassy phase is extremely slow and has very strong memory effects.

Another hallmark of finite dimensional disordered quantum spin models are Griffiths-McCoy singularities, that lead to a highly non trivial paramagnetic phase and critical behaviour.

In this contribution we summarize the results of recent studies of the statics, dynamics and critical properties of mean-field and one dimensional quantum disordered spin systems coupled to an environment.<sup>7</sup> We also briefly mention related studies on a driven mesoscopic ring,<sup>17</sup> dilute antiferromagnets,<sup>18</sup> and a manifold in an infinite dimensional quenched random potential.<sup>19</sup> In Sect. 2 we review the main questions that we attempted to answer in these papers. In Sect. 3 we recall the definition of the models that we studied. The basic techniques used to study classical glassy models with or without disorder are well documented in the literature (the replica trick, scaling arguments and droplet theories, the dynamic functional method used to derive macroscopic equations from the microscopic Langevin dynamics, functional renormalization, Montecarlo and molecular dynamic numerical methods). On the contrary, the techniques needed to deal with the statics and dynamics of quantum macroscopic systems are much less known in general. We briefly mention the ones that we used to study dissipative disordered quantum models.<sup>20</sup> Finally, in Sect. 4 we list some projects for future research.

## 2. Questions and main results

### 2.1. *Effect of quantum fluctuations on glassiness*

Since glasses are not expected to reach equilibrium in experimentally accessible times, it is important to devise a method to understand the influence of quantum fluctuations on their truly nonequilibrium *real-time* dynamics. Intuitively, one expects quantum fluctuations to only affect the short-time dynamics; however, they

are also expected to act as thermal fluctuations. It is then not clear *a priori* whether quantum fluctuations would tend to destroy glassiness or modify it drastically.

The usual methods of equilibrium quantum statistical mechanics are inappropriate to describe this nonstationary situation. We presented a formalism suited to study the real-time dynamics of a general nonlinear, possibly disordered, model in contact with a bath that can also be applied to glassy models.<sup>11</sup> The method is a combination of the Schwinger-Keldysh or closed-time path technique to study real-time phenomena, with the Feynman-Vernon approach to dissipation that consists in modelling the coupling to the environment with an ensemble of quantum harmonic oscillators. As a particular case we studied the relaxation of the spherical version of the quantum  $p$  spin fully-connected disordered model. We analyzed the relaxation of ‘random initial conditions’ in the limit of vanishing coupling strength taken after the long waiting-time ( $t_w$ ) limit. The same technique was applied to other quantum problems<sup>21–25</sup> and related studies appeared.<sup>26–30</sup> We later studied the effect of a strong coupling to the environment<sup>14</sup> (*i.e.*  $t_w \rightarrow \infty$  with  $\alpha$  finite) as discussed below.

In the disordered phase the dynamics is fast and occurs in equilibrium. The correlation and linear response are stationary, *i.e.*  $C(t + t_w, t_w) = C(t)$  and  $R(t + t_w, t_w) = R(t)$ . They both oscillate with a  $\Gamma$ -dependent frequency that also depends on the characteristics of the bath ( $\alpha, s$ ) if  $\alpha$  is not taken to zero. Correlations and responses are linked by the quantum fluctuation dissipation theorem (FDT). At high temperatures and after a short transient the system decoheres and the dynamics becomes classical (*e.g.* responses and correlations are related by the classical FDT).

In the ordered phase the glassy dynamics persists asymptotically if the thermodynamic limit has been taken at the outset. The dynamics of glassy systems occurs out of equilibrium and the correlations and responses loose time translation invariance. If  $t_w$  denotes the time elapsed since a quench into the SG phase,  $C(t + t_w, t_w)$  and  $R(t + t_w, t_w)$  depend on both  $t$  and  $t_w$ . The order in which the limits  $t_w \rightarrow \infty$  and  $t \rightarrow \infty$  are taken is very important. For sufficiently long  $t$  and  $t_w$  but in the regime  $t \ll t_w$ , the dynamics is stationary and the correlation reaches a plateau  $q_{ea}$ . A few quantum oscillations exist at very short  $t$  (and arbitrarily large value of  $t_w$ ) and they later disappear. For times  $t > t_w$ , the system enters an *aging* regime where the correlation function depends on  $t_w$  explicitly. In this regime, the correlation vanishes at long times,  $\lim_{t \rightarrow \infty} C(t + t_w, t_w) = 0$ , at a rate that depends on  $t_w$ . The behaviour is thus qualitative similar to what observed classically, even if the scaling laws are modified by the quantum fluctuations. One checks that the relaxation of typical (highly energetical) initial conditions approaches a threshold level in phase space with a higher energy-density than the equilibrium one.

The comparison of responses and correlations in the ordered phase is particularly interesting. The quantum FDT is a complicated integral relation between the correlation and the linear response. This relation holds for  $t \ll t_w$  when the correlation decays to the plateau. In the second regime the quantum FDT is no longer verified, much as it happens in the classical problem. A comparison of the integrated

responses and the symmetrized correlation in a parametric manner<sup>8</sup> shows that the two quantities are related by a *classical* FDT with an effective temperature<sup>31</sup>  $T_{eff}$  that depends on the parameters in the problem  $(T, \Gamma)$  [and the characteristics of the bath,  $\alpha, s$ ]. We proved that  $T_{eff} > T$  and it is different from zero even when the environment is at  $T = 0$ .  $T_{eff}$  drives the dynamics at late epochs and it makes the dynamics appear classical in that two-time regime.  $T_{eff} > 0$  even when the temperature of the bath is zero. The generation of a non-trivial  $T_{eff}$  for the slow part of the decay gives support to the “decoherent” effect observed in the decay of correlations.

The similarity between the second decay in the classical and quantum problem can be argued as follows. The responses decay rather fast to zero when the time difference increases (though integrated over a time-interval of very long length does not vanish.<sup>36</sup>) The explicit dependence on  $\hbar$  in the regime of widely separated times comes from factors with higher powers of  $R$  that vanish<sup>11</sup>. The effect of quantum fluctuations on the slow time-difference regime is simply to renormalize<sup>22</sup> certain parameters in the equations that otherwise look classical.

Models with  $p \geq 3$  interactions have different static and dynamic phase transitions, with the latter surrounding a larger region of the  $(T, \Gamma)$  phase diagram. The static and dynamic ordered-disorder phase transition present a second-to-first order transition.<sup>53,12</sup> Close to the classical critical temperature quantum effects are small and the phase transition is discontinuous but of second order, as in the classical case. There are no discontinuities in the thermodynamic quantities but there is a plateau that develops in the correlation function when the transition is approached from the disordered side. This is the behaviour expected in classical glasses. Spin-glasses instead have continuous transitions (without precursors). Conversely, close to the quantum critical point quantum fluctuations drive the transition first order thermodynamically. Across the first order line the susceptibility is discontinuous and shows hysteresis. This is similar to what has been observed in the dipolar-coupled Ising magnet  $\text{LiHo}_x\text{Y}_{1-x}\text{F}_4$  in a transverse field.<sup>4</sup>

We also adapted the *Ansatz* of marginal stability<sup>12,50,53</sup> to identify the dynamic critical line that is consistent with the one found using the Schwinger-Keldysh formalism. The analytic continuation of the imaginary-time dependent correlation computed with the AMS in the absence of the bath is identical to the *stationary* part of the non-equilibrium correlation function ( $C > q_{ea}$ ) when one takes the long-time limit first and the limit in which the coupling to the bath goes to zero next.<sup>12,19</sup>

In the classical case, the study of the Thouless-Anderson-Palmer (TAP) free energy landscape has been very useful to understand the behaviour of these systems.<sup>8</sup> A TAP approach can also be developed for quantum problems.<sup>13</sup> It helps understanding the existence of a dynamic and a static critical line as well as the change in nature of the transition close to the quantum critical point.

## 2.2. Effect of the bath: decoherence and localization

The quantum systems mentioned in Sect. 1 are not totally isolated but in contact with environments of different type.

The low-energy physics of many tunneling systems is well described by the spin-boson model in which the two equivalent degenerate states are represented by the eigenstates  $\sigma_z = \pm 1$  of an Ising pseudo-spin. A transverse field coupled to  $\sigma_x$  (say) represents the tunneling matrix element. The coupling to the environment is given in terms of its spectral density  $I(\omega) \propto \alpha \omega^s$  for  $\omega \ll \omega_c$ , where  $\alpha$  is a dimensionless coupling constant and  $\omega_c$  a high frequency cutoff. The exponent  $s$  characterizes different types of environment. The Ohmic case ( $s = 1$ ) is quite generally encountered but superOhmic ( $s > 1$ ) and subOhmic ( $s < 1$ ) baths also occur in, *e.g.*, the Kondo effect in unconventional hosts.

The coupling of quantum two-level systems (TLS) to a dissipative environment has decisive effects on their dynamical properties. The dilute case, in which interactions between the TLS can be neglected, has been extensively investigated.<sup>32,33</sup> This problem, is related the  $1d$  Ising model with inverse squared interactions and the anisotropic Kondo model. In the Ohmic case, at zero temperature, there is a phase transition<sup>34</sup> at  $\alpha = 1$ . For  $\alpha < 1$  there is tunneling and two distinct regimes develop. If  $\alpha < 1/2$  the system relaxes with damped coherent oscillations; in the intermediate region  $1/2 < \alpha < 1$  the system relaxes incoherently. For  $\alpha > 1$  quantum tunneling is suppressed and  $\langle \hat{\sigma}_z \rangle \neq 0$  signalling that the system remains localized in the state in which it was prepared. These results also hold for sub-Ohmic baths while weakly damped oscillations persist for super-Ohmic baths. At finite temperatures (but low enough such that thermal activation can be neglected), there is no localization but the probability of finding the system in the state it was prepared decreases slowly with time for  $\alpha > \alpha^c$ .

The effect of dissipation on the phase transition, critical behaviour, ordered phase, localization and decoherence properties of macroscopic interacting systems is only now starting to be analyzed.<sup>14,15,35</sup> In thermodynamic equilibrium, in the absence of the bath, the interactions between the TLS lead to the appearance of an ordered state at low enough temperature. If the interactions are of random sign, as in the models we considered, the latter will be a glassy state. In this phase the symmetry between the states  $\sigma_i^z = \pm 1$  at any particular site is broken but there is no global magnetization,  $\sum_i \langle \hat{\sigma}_i^z \rangle = 0$ . The coupling to the bath also competes with the tunneling term. We thus expect the presence of noise to increase the stability of the glassy state. The consequences of this fact are particularly interesting when there is localization at some  $\alpha = \alpha^c$ : a quantum critical point at  $J = 0$ ,  $\alpha = \alpha^c$  separates the disordered and the ordered state such that, for  $\alpha > \alpha^c$ , the glassy phase survives down to  $J = 0$ .

A system of non-interacting localized TLS and a glassy state *in equilibrium* are in some way similar. However, this resemblance is only superficial. The details of the dynamics of the two systems are expected to be quite different, with  $C$  saturating

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at a finite value in the localized state, and  $C$  decaying down to zero in the glassy phase.

### 2.3. *Effect of the bath on interacting mean-field models*

The problem of a single TLS being a difficult one, that of an infinite set of interacting TLS seems hardly solvable. Therefore, as a first step, we focused on the effect of the reservoir on the  $p$ -spin *spherical* model,<sup>14</sup> a problem that we studied with the real-time approach and the replica Matsubara technique. The position of the critical lines strongly depends on the strength of the coupling to the bath and the type of bath (Ohmic, subOhmic, superOhmic). For a given type of bath, the ordered glassy phase is favored by a stronger coupling. The classical static and dynamic critical temperatures remain unchanged by the coupling to the environment. The identity between the analytic continuation of the imaginary-time correlation to real-time and the correlation in the Schwinger-Keldysh approach also holds if the strength of the bath is finite.

The spherical model localizes in the absence of interactions when coupled to a subOhmic bath:  $C(t + t_w, t_w)$  reaches, for any waiting-time  $t_w$  and long enough  $t$ , a plateau that it never leaves. When interactions are switched on localization disappears and the system undergoes a phase transition towards a glassy phase.

Similar results were found for the  $SU(N)$  random Heisenberg model in the limit  $N \rightarrow \infty$ <sup>23</sup>, the  $p = 2$  spherical model<sup>25</sup> and the  $SU(2)$   $p$ -spin model.<sup>15</sup>

### 2.4. *Effect of disorder: Griffiths singularities*

Griffiths singularities<sup>37–39</sup> in classical finite dimensional random systems are so weak that their consequences have not been clearly observed neither experimentally nor numerically. Instead, when quantum fluctuations are introduced they are much stronger and rare regions completely determine the static and dynamic behaviour in the Griffiths phase.

The isolated quantum random Ising chain has been studied in great detail with a decimation technique.<sup>40</sup> It undergoes a quantum phase transition from a PM to a ferromagnet for a special relation between the distribution of exchanges and transverse fields. The quantum phase transition is of second order with the correlation time scaling exponentially with the spatial correlation length (*activated* scaling). Within the renormalization group procedure this is a characteristic of an “infinitely strong disorder” fixed point.<sup>40</sup> Within the PM Griffiths phase the distributions of local linear and non-linear magnetic susceptibilities are large and typical and average values are very different, with the latter being dominated by rare regions. The behaviour in higher spatial dimensions is similar.

The analysis of Montecarlo simulations of the equivalent  $d + 1$  classical Ising model is quite tricky. Initially, it was claimed that there was conventional scaling in  $d = 1$ <sup>41</sup> as well as in  $d > 1$ <sup>42</sup> but a more careful analysis of the numerical data confirmed the activated scaling in both cases.<sup>43</sup>

On the real-time dynamic side, there have been some studies of the relaxation of special initial conditions of the isolated random Ising chain at constant energy.<sup>44</sup>

### 2.5. Disorder and dissipation: fate of Griffiths phase?

If the interactions between two-level systems placed in a finite dimensional space are random one may wonder what is the effect of the bath on the Griffiths phase. The answer to this question has been debated over the last years.<sup>45–48</sup> We addressed this problem using Montecarlo simulations of the equivalent  $2d$  classical system.<sup>16</sup> Preliminary results for rather small systems ( $N_x \leq 32$ ,  $N_\tau \leq 256$ ) show that an Ohmic bath favors the glassy phase. Our results are compatible with (but we do not prove) activated scaling at criticality, at least for small values of  $\alpha$ .

## 3. The models

Disordered quantum spin- $\frac{1}{2}$  models with two-body interactions are defined by

$$H_S = - \sum_{ij} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + \sum_i \Gamma_i \hat{\sigma}_i^x + \sum_i h_i \hat{\sigma}_i^z. \quad (1)$$

$i = 1, \dots, N$  labels the spins that lie on the vertices of a cubic  $d$  dimensional lattice and are represented by Pauli matrices. The interaction strengths  $J_{ij}$  couple near-neighbours only and are chosen from a probability distribution,  $P(J)$ . The average and variance are defined as  $[J_{ij}] = J_o$  and  $[J_{ij}^2] = J^2/(2c)$ , where  $J_o$  and  $J$  are  $O(1)$  and  $c = 2d$  is the connectivity of the lattice. The next-to-last term is a coupling to a random quenched local transverse field  $\Gamma_i$ . The last term is the coupling to a longitudinal field that serves to compute local susceptibilities.

Several generalizations that render the model easier to treat analytically are:

- *Fully-connected limit.* One allows each spin to interact with all others,  $c \rightarrow N - 1$ .
- *Multi-spin interactions.* In the fully-connected case one can consider

$$H_S = - \sum_{i_1 \dots i_p} J_{i_1 \dots i_p} \hat{\sigma}_{i_1}^z \dots \hat{\sigma}_{i_p}^z + \sum_i \Gamma_i \hat{\sigma}_i^x + \sum_i h_i \hat{\sigma}_i^z. \quad (2)$$

where the sum runs over all  $p$ -uplets,<sup>52,53,12,15</sup> with  $p$  an integer parameter,  $p \geq 2$ . The exchanges are random independent variables with variance  $p!J^2/(2N^{p-1})$ . This model provides a mean-field description of the structural glass transition and glassy physics that is also intimately related to the mode-coupling approach.<sup>8</sup> In its dilute limit, with no geometry but finite connectivity on each site, this model is related to the  $K$ -sat optimization problem.<sup>56</sup>

– *Spherical variables – a particle in a random potential.* One considers the spherical limit,  $\sum_{i=1}^N \langle \hat{\sigma}_i^2 \rangle = N$ , in which the  $\hat{\sigma}_i$  may be interpreted as the coordinates of a particle moving on an  $N$ -dimensional sphere. A kinetic term,  $K = \sum_{i=1}^N \hat{P}_i^2/(2M)$ , is then included in the Hamiltonian, with  $\hat{P}_i$  the conjugated momentum satisfying the commutation rules  $[\hat{P}_i, \hat{P}_j] = 0$ ,  $[\hat{P}_i, \hat{\sigma}_j] = -i\hbar\delta_{ij}$ . Other spherical models have been discussed.<sup>49</sup>

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The coupling to the environment is modelled by  $H = H_S + H_B + H_I + H_{CT}$ , where  $H_B$  is the Hamiltonian of the bath,  $H_I$  represents the interaction between the system and the bath and  $H_{CT} = \sum_{l=1}^{\tilde{N}} (2m_l \omega_l^2)^{-1} (\sum_{i=1}^N c_{il} \hat{\sigma}_i^z)^2$  is a counter-term that is usually added to eliminate an undesired mass renormalization induced by the coupling.<sup>32</sup> We assume that each spin is coupled to its own set of  $\tilde{N}/N$  independent harmonic oscillators with  $\tilde{N}$  the total number of them. For simplicity we consider the bilinear coupling,  $H_I = -\sum_{i=1}^N \hat{\sigma}_i^z \sum_{l=1}^{\tilde{N}} c_{il} \hat{x}_l$ . For  $p = 2$  the fully-connected limit with two-body interactions models metallic spin-glasses.<sup>51</sup>

#### 4. Perspectives

In this article we reviewed recent studies of insulating disordered magnets.

The principal merit of the fully-connected models is that they are simple enough to be studied in detail. Yet, many of their properties are generic and expected to hold at least qualitatively for more realistic cases.<sup>8</sup> The analysis of the fully-connected models is by now quite complete, having applied the replica theory, the real-time dynamics approach, and the investigation of the TAP free-energy landscape. A problem that remains not fully developed though is the treatment of the relaxation of initial conditions that are correlated with disorder.<sup>55</sup>

Recently, much progress has been done in the study of the statics of classical dilute spin models, that is to say, models defined on random hyper graphs with finite connectivity. The statics of these models encode problems in combinatorial optimization such as  $K$ -sat.<sup>56</sup> An interesting mapping relates (isolated) dilute quantum disordered spin systems to the dynamics of special purpose algorithms used in combinatorial optimization – such as Walk-Sat.<sup>57</sup> It would be interesting to study the latter using tools developed for the former and *vice versa*.

The great challenge remains to understand the behaviour of glassy systems with and without disorder in finite dimensions. In particular, one could try to adapt the decimation technique<sup>40</sup> to study disordered spin- $\frac{1}{2}$  models in contact with an environment. One can also envisage applications of the ideas described in this paper to other quantum systems evolving out of equilibrium. In this respect, we have studied the generation of an effective temperature in a conducting ring threaded by time-dependent magnetic field and coupled to a reservoir;<sup>17</sup> and we plan to analyse quantum dilute antiferromagnets<sup>18</sup> and the low-temperature dynamics of the Bragg glass,<sup>19</sup> as well as other related physical problems.

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